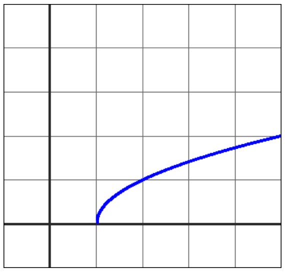
Calculus 140, section 2.4 One-Sided and Infinite Limits

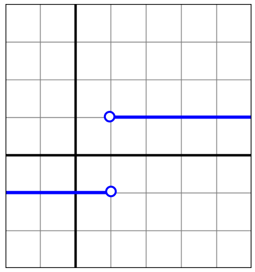
notes prepared by Tim Pilachowski

Examples A–D: Consider the following functions. Why is it problematic to try to evaluate $\lim_{x \to 1} f(x)$ for them?

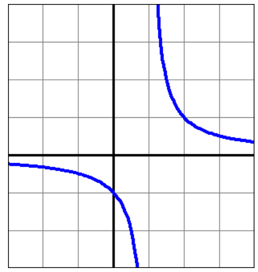
A) $f(x) = \sqrt{x-1}$



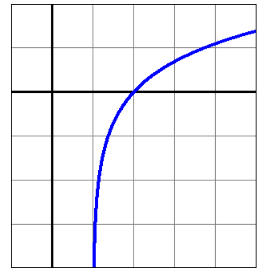
B) $f(x) = \dfrac{|x-1|}{x-1}$



C) $f(x) = \dfrac{1}{x-1}$



D) $f(x) = \ln(x-1)$



Definition 2.4: "Let f be a function defined at each point of some open interval (c, a). A number L is the **limit of $f(x)$ as x approaches a from the left** (or is the **left-hand limit of f at a**) if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a - \delta < x < a, \text{ then } |f(x) - L| < \varepsilon.$$

In this case we write $\lim_{x \to a^-} f(x) = L$ and say that the **left-hand limit of f at a** exists."

The notation $\lim_{x \to a^-} f(x)$ is read "the limit of $f(x)$ as x approaches a from the left".

If we were to consider some open interval (a, c) to the *right* of a, we get the analogous **right-hand limit of f at a**. If $\lim_{x \to a^+} f(x) = L$ we say that the **right-hand limit of f at a** exists."

The notation $\lim_{x \to a^+} f(x)$ is read "the limit of $f(x)$ as x approaches a from the right".

How do these **one-sided limits** connect to the ordinary, or **two-sided limits** of section 2.2?

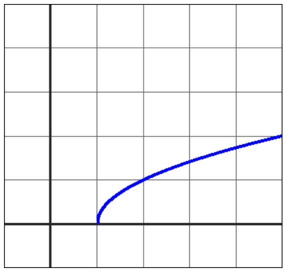
Theorem 2.5 (short version): If both one-sided limits exist and also $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x)$ exists, and

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

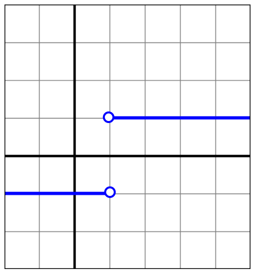
Good news: All of the properties given in Lecture 2.3 (sum rule, constant multiple rule, etc.) apply to one-sided limits!

As always, you should read through the more detailed explanations in the text, and look over the text's worked-out Examples.

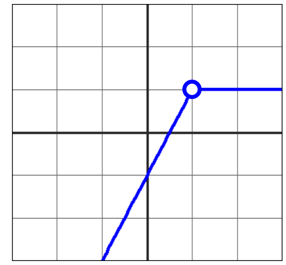
Example A: Given $f(x)=\sqrt{x-1}$, evaluate $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$.



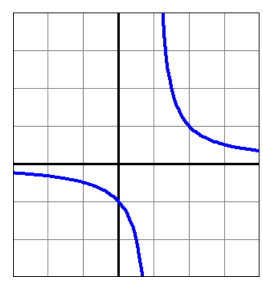
Example B: Given $f(x)=\frac{|x-1|}{x-1}$, evaluate $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$.



Example B extended: Given $f(x) = \begin{cases} 2x-1 & \text{for } x<1 \\ 1 & \text{for } x>1 \end{cases}$, evaluate $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1} f(x)$.

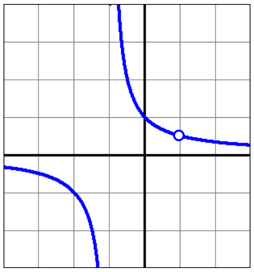


Example C: Given $f(x) = \dfrac{1}{x-1}$, evaluate $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1} f(x)$.

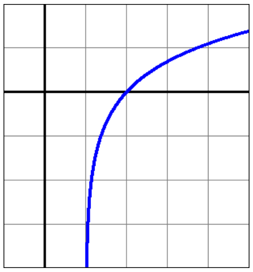


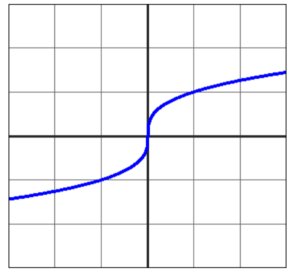
Definition 2.6 (short version): If $\lim\limits_{x \to a^{+\text{or}-}} f(x) = \infty$, or $\lim\limits_{x \to a^{+\text{or}-}} f(x) = -\infty$, then "the vertical line $x = a$ is called a **vertical asymptote of the graph of *f***, and we say that we say that f has an **infinite** … **limit at *a***."

Example C extended: Given $f(x) = \dfrac{x-1}{x^2-1}$, find all vertical asymptotes.



Example D: Given $f(x) = \ln(x - 1)$, evaluate $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1} f(x)$.





The text considers the graph of $f(x) = \sqrt[3]{x}$. While it has no vertical *asymptotes*, something interesting occurs when we consider the slope of the *tangent* at $x = 0$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{x^{1/3} - 0}{x - 0} = \frac{1}{x^{2/3}} = \infty$$

Definition 2.7: "Suppose *f* is continuous at *a*. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \infty$ or $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = -\infty$ then we say that the graph of *f* has a **vertical tangent at (*a*, *f*(*a*))**. In that case the vertical line $x = a$ is called the **line tangent to the graph of *f* at *a*.**"